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ALMOST PERIODIC SOLUTIONS OF THE KdV EQUATION*

Dedicated to Joachim Weyl, on the occasion of his 60th birthday, and in recognition of his role in nurturing applied mathematics. Through his influence on science policy and his eloquent advocacy of applications, he has encouraged many young mathematicians in the United States to choose secular rather than monastic mathematics.

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Abstract. In this talk we discuss the almost periodic behavior in time of space periodic solutions of the KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

We present a new proof, based on a recursion relation of Lenart, for the existence of an infinite sequence of conserved functionals $F_n(u)$ of form $\int P_n(u)\,dx$, P_n a polynomial in u and its derivatives; the existence of such functionals is due to Kruskal, Zabusky, Miura and Gardner. We review and extend the following result of the speaker: the functions u minimizing $F_{N+1}(u)$ subject to the constraints $F_j(u) = A_j$, $j = 0, \cdots, N$, form N-dimensional tori which are invariant under the KdV flow. The extension consists of showing that for certain ranges of the constraining parameters A_j the functional $F_{N+1}(u)$ has minimax stationary points; these too form invariant N-tori. The Hamiltonian structure of the KdV equation, discovered by Gardner and also by Faddeev and Zakharov, which is used in these studies, is described briefly. In an Appendix, M. Hyman describes numerical studies of the stability of some invariant 2-tori for the KdV flow; the numerical evidence points to stability.

1. Introduction. A recent series of investigations of nonlinear wave motion, commencing with Kruskal and Zabusky's paper [29], have led to the unexpected discovery that an astonishingly large number of important differential equations of mathematical physics are completely integrable Hamiltonian systems. Included among these are the Korteweg–de Vries (KdV) and Boussinesque equations for waves in shallow water, the equations governing self-induced transparency, and self-focusing and self-modulating waves in optics, the vibrations of the Toda lattice, the motion of particles under an inverse square potential, and some others.

These equations have been studied under two kinds of boundary conditions:

(i) Solutions are required to be periodic in space.

(ii) Solutions propagate in free space but are required to vanish at ∞.

We shall call (i) the compact, (ii) the noncompact case. It turns out that solutions behave quite differently in the two cases. In terms of Hamiltonian theory the difference can be explained in the following way:

A Hamiltonian system

$$\frac{dq_j}{dt} = H_{p_j}, \qquad \frac{dp_j}{dt} = -H_{q_j} \tag{1.1}$$

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is completely integrable if there is a canonical transformation introducing new conjugate variables $\bar{p}_j$, $\bar{q}_j$ and a new Hamiltonian $\bar{H}$ so that $\bar{H}$ is a function of $\bar{p}$ alone and is independent of $\bar{q}$. The Hamiltonian system in these new variables is

$$\frac{d\bar{p}_j}{dt} = -H_{\bar{q}_j} = 0$$

which implies that each $\bar{p}_j$ is constant; therefore

$$\frac{d\bar{q}_j}{dt} = H_{\bar{p}_j} = \text{const.},$$

so that

$$\bar{q}_j(t) = \bar{q}_j(0) + tH_{\bar{p}_j}. \tag{1.2}$$

The difference between the compact and noncompact case is this: in the compact case the $\bar{q}_j$ are angle variables, i.e., the original variables are periodic functions of the $\bar{q}_j$, whereas in the noncompact case there is no such periodicity. When we express the original coordinates q_j and p_j in terms of $\bar{q}_j$ and $\bar{p}_j$, we see that in the compact case every flow is a function of periodic motions; since the periods are, in general, incommensurable, we see that flows in the compact case are *almost periodic*. In the noncompact case, on the other hand, the time dependence in (1.2) represents a genuine linear growth which describes the manner in which particles or waves tend to infinity.

Starting with the work of Gardner, Greene, Kruskal and Miura [10], Faddeev and Zakharov [6] have shown that the KdV equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.3}$$

constitutes, for solutions defined on the whole real axis and zero at $x = \pm\infty$, a completely integrable Hamiltonian system, whose action and angle variables are simply related to the so-called *scattering data* of the associated Schrödinger operator, see (3.22). For solutions which are periodic with respect to x no such formulas are known; nevertheless it is strongly suspected that in this case too we are dealing with a completely integrable Hamiltonian system; the following items are evidence for this:

1. numerical calculations by Kruskal and Zabusky which indicate that solutions of KdV which are periodic in x are almost periodic in t, [29];
2. the construction of infinitely many conserved quantities by Gardner, Kruskal and Miura;
3. Gardner's observation that these functionals are in involution;
4. the existence of compact submanifolds of arbitrary finite dimension which are invariant under the KdV equation.

In this talk we review these facts:

In § 2 we describe a relation between conserved functionals and invariant submanifolds of flows. In § 3 we give a new proof of the existence of infinitely many conserved functionals, and show that they are in involution. In § 4 we display the Hamiltonian structure of the KdV equation; in § 5 we describe the construction with the aid of a minimum problem of invariant submanifolds which are N-dimensional tori, and on which solutions of KdV are almost periodic in t. In an Appendix, M. Hyman describes a calculation of some invariant 2-dimensional tori, and demonstrates by means of another calculation the remarkable stability of these manifolds.

At the end of § 5 we give an indication how invariant tori might be constructed with the aid of a minimax problem.

2. Equations of evolution, invariant submanifolds and conserved functionals. We consider equations of evolution of the form

$$\frac{d}{dt}u = K(u), \tag{2.1}$$

K an operator, in general nonlinear, mapping a linear space into itself. We assume that the initial value problem is properly posed, i.e., that solutions of (2.1) are uniquely determined by their values at $t=0$, that the initial value of u can be prescribed arbitrarily, and that solutions exist for all t. The mapping of initial data of solutions of (2.1) into data at time t can be thought of as a *flow*.

Let $F(u)$ be a *functional*, i.e., a numerically-valued function, in general nonlinear, defined on the underlying linear space. F is called differentiable if the directional derivative

$$\lim_{\varepsilon\to 0}\frac{F(u+\varepsilon v)-F(u)}{\varepsilon}$$

exists for all u and v and is a linear functional of v. We assume that our linear space is equipped with a *scalar product* $(\cdot,\cdot)$; since linear functionals can be expressed as scalar products, we can write

$$\frac{d}{d\varepsilon}F(u+\varepsilon v)\bigg|_{\varepsilon=0} = (G_F(u), v). \tag{2.2}$$

$G_F(u)$ is called the *gradient* of F at u with respect to the specified scalar product. G_F is a nonlinear operator.

Let F be a functional for which $F(u(t))$ is independent of t for all solutions of (2.1). Such an F is called a *conserved functional* of the flow.

Differentiating, using the definition of gradient and equation (2.1) gives

$$\frac{d}{dt}F(u(t)) = (G_F(u), u_t) = (G_F(u), K(u)); \tag{2.3}$$

we deduce the following theorems from this.

THEOREM 2.1. *$F(u)$ is an invariant functional of* (2.1) *if for all u,*

$$(G_F(u), K(u)) = 0.$$

THEOREM 2.2. *Let F be a differentiable conserved functional of the flow* (2.1). *Then solutions of*

$$G_F(u) = 0 \tag{2.4}$$

form an invariant manifold of the flow (2.1).

A formal proof of Theorem 2.2 is given in [18]; here is a simple intuitive argument:

Solutions of (2.4) are stationary points of $F(u)$, i.e., points u_0 such that for every smooth curve $u(\varepsilon)$ issuing from $u_0 = u(0)$,

$$\frac{d}{d\varepsilon} F(u(\varepsilon))\Big|_{\varepsilon=0} = 0. \tag{2.5}$$

Since the flow carries smooth curves into smooth curves, and conserves the values of F, (2.5) is true at all points along the trajectory issuing from u_o.

3. Conserved functionals of the KdV equation. The usefulness of Theorem 2.2 depends on the existence of many conserved functionals whose gradients have tractable null sets. The KdV equation is rich in such functionals; three of them are classical:

$$\begin{aligned} F_0(u) &= \int 3u\,dx, \\ F_1(u) &= \int \tfrac{1}{2}u^2\,dx, \\ F_2(u) &= \int (\tfrac{1}{6}u^3 - \tfrac{1}{2}u_x^2)\,dx. \end{aligned} \tag{3.1}$$

The gradients of these functionals are

$$\begin{aligned} G_0 &= 3, \\ G_1 &= u, \\ G_2 &= \tfrac{1}{2}u^2 + u_{xx}. \end{aligned} \tag{3.2}$$

To prove that the functionals (3.1) are conserved for KdV we have to verify condition (2.3) of Theorem 2.1, with G_F given by (3.2) and $K(u) = -u_{xxx} - uu_x$.

Kruskal and Zabusky made the remarkable discovery that there are further conserved functionals, of which

$$F_3(u) = \int (\tfrac{5}{72}u^4 - \tfrac{5}{6}uu_x^2 + \tfrac{1}{2}u_{xx}^2)\,dx \tag{3.1$_3$}$$

is the first example. Eventually Gardner, Kruskal and Miura proved, see [11], that these four are the beginning of an infinite sequence of conserved functionals F_n of the form

$$F_n(u) = \int P_n\,dx, \tag{3.1$_n$}$$

P_n a polynomial in u and its derivatives up to order $n-1$. We give now a new proof of the existence of these functionals F_n, based on a remarkable recursion formula for their gradients discovered by Andrew Lenart some years ago, first published in [11]:

$$HG_n = \partial G_{n+1}, \tag{3.3}$$

where H is the third order operator

$$H=\partial^3+\tfrac{2}{3}u\partial+\tfrac{1}{3}u_x, \qquad \partial=d/dx; \tag{3.4}$$

note that H is antisymmetric:

$$H^*=-H. \tag{3.5}$$

It is easy to verify by a calculation that (3.3) holds for $n=0$, 1 and 2. Next we show the following theorem.

THEOREM 3.1. *There exists a sequence G_n of polynomials in u and its derivatives up to order $2n-2$ which satisfy* (3.3); *G_n is uniquely determined if we set the constant term equal to zero.*

Proof. We use the following simple calculus lemma: Suppose that Q is a polynomial in derivatives of u up to order j, such that for every periodic function u of period p

$$\int_0^p Q(u)\,dx=0.$$

Then there exists a polynomial G in derivatives of u up to order $j-1$ such that

$$Q=\partial J.$$

We assume that G_j has been constructed for all $j\leqq n$; to construct G_{n+1} we have to solve (3.3). According to the above calculus lemma, we have to show that for all u,

$$\int HG_n\,dx=0.$$

Since according to $(3.2)_0$, $G_0=3$, we can rewrite this equation as

$$(HG_n, G_0)=0. \tag{3.6}$$

Using repeatedly the antisymmetry of H and ∂ and relation (3.3) we can write the following sequence of identities:

$$\begin{aligned}(HG_n, G_0)&=-(G_n, HG_0)=-(G_n, \partial G_1)\\ &=(\partial G_n, G_1)=(HG_{n-1}, G_1)=\cdots\\ &=\begin{cases}(HG_{n/2}, G_{n/2}) & \text{if } n \text{ is even},\\ (\partial G_{(n+1)/2}, G_{(n+1)/2}) & \text{if } n \text{ is odd}.\end{cases}\end{aligned}$$

Because H and ∂ are antisymmetric, both of these expressions are zero; this proves that the compatibility relation (3.6) is fulfilled and completes the proof of Theorem 3.1.

The argument presented above can be used, with trivial alterations, to prove the following result of Gardner [9] which plays an important role in the Hamiltonian theory of the KdV equation.

THEOREM 3.2. *For all m and n and all u,*

$$(G_m, \partial G_n)=0. \tag{3.7}$$

Next we show the following theorem.

THEOREM 3.3. *G_n is the gradient of a functional of form* $(3.1)_n$.

Proof. Just as in finite-dimensional spaces, gradients G are characterized by the *symmetry* of their derivatives. Set

$$\frac{d}{d\varepsilon} G(u+\varepsilon v)\bigg|_{\varepsilon=0} = N(u)v. \tag{3.8}$$

Suppose G is the gradient of F; then

$$\frac{d^2}{d\varepsilon\, d\eta} F(u+\varepsilon v+\eta w)\bigg|_{\varepsilon=\eta=0}$$

is equal to

$$(N(u)w, u) \quad \text{or} \quad (N(u)v, w)$$

depending on whether we differentiate first with respect to ε or η. Since mixed partials are equal, the symmetry of $N(u)$ follows.

To prove the converse, take a smooth path $u(\varepsilon)$ connecting u_0 and u_1; denote $(d/d\varepsilon)u$ by u_ε. Integrating

$$\frac{d}{d\varepsilon} F(u(\varepsilon)) = (G(u), u_\varepsilon)$$

with respect to ε we get

$$F(u_1) - F(u_0) = \int (G(u), u_\varepsilon)\, d\varepsilon. \tag{3.9}$$

Given G whose derivative is symmetric we define F by formula (3.9); to verify that G is the gradient of F so defined we have to show that definition (3.9) is independent of the path connecting u_0 and u_1. To verify this independence we consider one parameter families of curves $u = u(\varepsilon, \eta)$ with common endpoints u_0 and u_1. The derivative of the right side with respect to η is

$$\int (G(u), u_{\varepsilon\eta})\, d\varepsilon + \int (N(u)u_\eta, u_\varepsilon)\, d\varepsilon.$$

We integrate the first term by parts; there are no boundary terms since $u_\eta = 0$ at the endpoints, and so we get

$$-\int (N(u)u_\varepsilon, u_\eta)\, d\varepsilon + \int (N(u)u_\eta, u_\varepsilon)\, d\varepsilon.$$

Clearly this quantity is zero if N is a symmetric operator.

Next we compute the derivative of G_n; let $u(\varepsilon)$ be a smooth curve, and denote differentiation with respect to ε by prime. Differentiating (3.3) with respect to ε we get

$$H'G_n + HG_n' = \partial G_{n+1}'. \tag{3.10}$$

Using the definition (3.4) of H we have

$$H'G_n = \tfrac{2}{3}u'G_{n_x} + \tfrac{1}{3}u_x'G_n$$

which can be rewritten as $K_n u'$, where K_n is the operator

$$K_n = \tfrac{1}{3}G_n\partial + \tfrac{2}{3}G_{n_x}. \tag{3.11}$$

Denote the derivative of G_n by N_n; by definition of derivative

$$G_n' = N_n u', \qquad G_{n+1}' = N_{n+1} u';$$

substituting this and the previous relation into (3.10) we get

$$K_n + HN_n = \partial N_{n+1}. \tag{3.12}$$

LEMMA 3.4.

$$K_{n-1}H - K_n \partial \tag{3.13}$$

is self-adjoint.

Proof. An explicit calculation using the definition (3.4) of H and $(3.11)_{n-1}$ of K_{n-1} gives the following formula:

$$3K_{n-1}H = G_{n-1}\partial^4 + 2G_{n-1_x}\partial^3 + \tfrac{2}{3}G_{n-1}u\partial^2 + (G_{n-1}u_x + \tfrac{4}{3}G_{n-1_x}u)\partial + \cdots.$$

A slightly more tedious calculation gives

$$3K_{n-1}H - 3(K_{n-1}H)^* = a_{n-1}\partial + b_{n-1}, \tag{3.14}$$

where

$$a_{n-1} = 2G_{n-1_{xxx}} + \tfrac{4}{3}G_{n-1_x} + \tfrac{2}{3}G_{n-1}u_x$$

and

$$b_{n-1} = G_{n-1_{xxxx}} + \tfrac{2}{3}G_{n-1_{xx}}u + G_{n-1_x}u_x + \tfrac{1}{2}G_{n-1}u_{XX}.$$

We observe, using (3.4), that

$$a_{n-1} = 2HG_{n-1}, \qquad b_{n-1} = \partial HG_{n-1}.$$

Using relation $(3.3)_{n-1}$ we get

$$a_{n-1} = 2G_{n_x}, \qquad b_{n-1} = G_{n_{xx}}.$$

Substituting these into (3.14) we get

$$K_{n-1}H - (K_{n-1}H)^* = \tfrac{2}{3}G_{n_x} + \tfrac{1}{3}G_{n_{xx}}. \tag{3.15}$$

On the other hand, a straightforward calculation gives

$$K_n\partial - (K_n\partial)^* = \tfrac{2}{3}G_{n_x} + \tfrac{1}{3}G_{n_{xx}}. \tag{3.16}$$

Subtracting (3.16) from (3.15) gives Lemma 3.4.

Now take equations $(3.12)_{n-1}$ and $(3.12)_n$:

$$K_{n-1} + HN_{n-1} = \partial N_n,$$
$$K_n + HN_n = \partial N_{n+1}.$$

We multiply the first equation by H on the right, the second by ∂ on the right, and subtract the first from the second; we get

$$\partial N_{n+1}\partial = K_n\partial - K_{n-1}H + \partial N_n H + HN_n\partial - HN_{n-1}H. \tag{3.17}$$

We assume as induction hypothesis that N_n and N_{n-1} are symmetric; since H is antisymmetric, it follows that the last term on the right is symmetric, and that the sum of the 3rd and 4th terms is symmetric. According to Lemma 3.4 the sum of the first two terms is symmetric; so we conclude from (3.17) that $\partial N_{n+1}\partial$ is symmetric.

From the symmetry of $\partial N_{n+1}\partial$ we conclude that N_{n+1} is symmetric on the subspace consisting of periodic functions whose mean value is zero. For such functions f_0 and g_0 can be written as derivatives of other periodic functions f and g; thus

$$(f_0, Ng_0) = (\partial f, N\,\partial g) = -(f, \partial N\,\partial g) = -(\partial N\,\partial f, g)$$
$$= (N\,\partial f, \partial g) = (Nf_0, g_0).$$

We claim that N_{n+1} is symmetric over the whole space; to show this we define F_{n+1} by formula (3.9) with $G = G_{n+1}$, taking $u_0 = 0$ and choosing as path of integration the straight line segments connecting $u_0 = 0$ to u. Since G_{n+1} is a polynomial in u and its derivatives, F_{n+1} is of the form

$$F_{n+1}(u) = \int P_{n+1}(u)\,dx, \tag{3.18}$$

P_{n+1} a polynomial in u and its derivatives. We claim that the gradient of F_{n+1} is G_{n+1}; denote the gradient of F_{n+1} by $\tilde{G}_{n+1}$. Since F_{n+1} is of form $(3.1)_{n+1}$, $\tilde{G}_{n+1}$ is a polynomial in u and its derivatives. N_{n+1} is symmetric on the subspace of functions with mean value zero, it follows that for any u and v in that subspace

$$\frac{d}{d\varepsilon} F_{n+1}(u+\varepsilon v)\bigg|_{\varepsilon=0} = (G_{n+1}(u), v).$$

Subtracting this from the definition of the gradient

$$\frac{d}{d\varepsilon} F_{n+1}(u+\varepsilon v) = (\tilde{G}_{n+1}(u), v),$$

we get that

$$(G_{n+1}(u) - \tilde{G}_{n+1}(u), v) = 0$$

for all v with mean value 0. This implies that

$$G_{n+1}(u) - \tilde{G}_{n+1}(u) = \text{const}.$$

We claim that this implies that $G_{n+1} = \tilde{G}_{n+1}$; for both G_{n+1} and $\tilde{G}_{n+1}$ are polynomials in u and its derivatives, without constant term. If they were not identical, one could easily construct a function of mean value zero such that $G_{n+1}(u) - \tilde{G}_{n+1}(u)$ is not constant. This completes the proof that G_{n+1} is a gradient; this implies that N_{n+1} is a symmetric operator, and the inductive step for the proof of Theorem 3.3 is complete.

THEOREM 3.5. *The functionals F_m are conserved for solutions of the* KdV *equation.*

Proof. Using formula $(3.2)_2$ for G_2 we see that the KdV equation can be rewritten in the form

$$u_t + \partial G_2(u) = 0, \tag{3.19}$$

i.e., $K(u) = -\partial G_2(u)$. Using Theorem 2.1 we conclude that F is a conserved functional of KdV if and only if

$$(G_F, \partial G_2) = 0. \tag{3.20}$$

According to relation (3.7) of Theorem 3.2, with $n=2$, the functionals F_m all satisfy this condition; therefore all F_m are conserved. This completes the proof of Theorem 3.5.

The same argument, when combined with the full force of Theorem 3.2 yields a more general result. We define the nth generalized KdV equation to be

$$u_t + \partial G_n(u) = 0. \tag{3.21$_n$}$$

The proof given above also serves to prove the following theorem.

THEOREM 3.6. *Each F_m is a conserved functional for all generalized* KdV *equations* (3.21).

We turn now to another class of conserved functionals. Gardner, Kruskal and Miura, see [11], have shown that the eigenvalues of the Schrödinger operator

$$L = \partial^2 + u/6 \tag{3.22}$$

are conserved functionals of the KdV equation. The author has shown that they are conserved functionals of all generalized KdV equations. Another proof of this has been given by Lenart, see [11]; here we present yet another proof.

Suppose λ is a simple eigenvalue of L:

$$Lw = \lambda w. \tag{3.23}$$

Then λ is a differentiable functional of the potential u occurring in L; the gradient of $\lambda(u)$ is easily computed by considering one parameter families $u(\varepsilon)$ and differentiating (3.22) with respect to ε. We get

$$Lw' + \frac{u'}{6} w = \lambda w' + \lambda' w,$$

where prime denotes derivative with respect to ε. Multiply this equation by u and integrate; using the symmetry of L and equation (3.23) we can eliminate w' and end up with this expression for λ':

$$\lambda' = \tfrac{1}{6} \int u' w^2 \, dx = \left(\frac{w^2}{6}, u' \right); \tag{3.24}$$

here we have assumed that w is normalized so that $(w, w) = 1$. By definition of the gradient G_λ,

$$\lambda' = (G_\lambda, u').$$

Comparing this with (3.24) we conclude that

$$G_\lambda = \tfrac{1}{6} w^2. \tag{3.25}$$

THEOREM 3.7. *Each λ is a conserved functional for all generalized* KdV *equations.*

Proof. As we saw earlier, for any functional F and any solution $u(t)$ of $(3.21)_m$,

$$\frac{dF(u(t))}{dt} = (G_F, u_t) = (G_F, \partial G_m),$$

so that F is conserved if and only if for all u,

$$(G_F, \partial G_m) = 0. \tag{3.26}$$

Applying this to $F = \lambda$ and using (3.25) to express the gradient of λ, we get

$$(w^2, \partial G_m) = 0 \tag{3.27}$$

as condition of invariance of λ. Next we make use of the following obscure but well-known lemma.

LEMMA 3.8 *Suppose w is an eigenfunction of L, satisfying* (3.23). *Then w^2 satisfies the differential equation*

$$Hw^2 = 4\lambda\, \partial w^2, \tag{3.28}$$

where H is defined by (3.4).

Remark. This relation can be verified by a simple calculation.

Irrelevant remark. The jth powers of the eigenfunctions satisfy a $(j+1)$st order equation.

By using the antisymmetry of H and ∂, equation (3.28) and the recursion relation (3.3) we get the following string of identities:

$$\begin{aligned}(w^2, \partial G_m) &= (w^2, HG_{m-1}) = -(Hw^2, G_{m-1})\\ &= -4\lambda(\partial w^2, G_{m-1}) = 4\lambda(w^2, \partial G_{m-1}) = \cdots\\ &= (4\lambda)^m (w^2, \partial G_0) = 0.\end{aligned}$$

In the last step we used the fact that by $(3.2)_0$, G_0 is a constant. This completes the proof of (3.27) and thus of Theorem 3.7.

Remark. In case of a double root we choose for the functional $F(u) = \lambda_1 + \lambda_2$, whose gradient is $w_1^2 + w_2^2$, where w_1, w_2 is any pair of orthonormal eigenvectors. The rest of the proof proceeds as before.

We return now to the Lenart recursion relation

$$HG_n = \partial G_{n+1} \tag{3.29}$$

which can be solved, starting with $G_0 = 1$, for all positive integers. These recursion relations can also be solved for negative integers n. Since $\partial G_0 = 0$, we have for $n = -1$,

$$HG_{-1} = 0.$$

We have shown in § 6 of [18] that this equation has nontrivial periodic solutions; here we offer a different, topological proof for this fact;

The operator H is antisymmetric; therefore its eigenvalues are purely imaginary. Since H is real, the eigenvalues are located symmetrically around the origin. Now consider a one-parameter family of functions $u(\varepsilon)$ entering H. The spectrum of $H(\varepsilon)$ is symmetric around the origin; therefore the multiplicity of 0 as eigenvalue changes by an even number. When $u = 0$, $H = \partial^3$; this operator has 0 as eigenvalue of multiplicity 1; so it follows from the previous argument that for any other u, H has 0 as eigenvalue with multiplicity 1 or 3.

Remark. This intuitive argument can easily be made rigorous, but it is hardly worthwhile to do so since the proof in [18] is perfectly straightforward.

Having shown that a nontrivial G_{-1} exists we can show that equation (3.3) has a solution G_n for $n=-2, -3, \cdots$. The compatibility relation in this case is that the right side, ∂G_{n+1}, be orthogonal to the nullspace of H. Since that nullspace is spanned by G_{-1} the condition is

$$(\partial G_{n+1}, G_{-1})=0.$$

This can be verified in the same manner as (3.7) was for Theorem 3.2. The only difference is that for n negative the G_n are no longer polynomials in u and their derivatives.

One could show that, if properly normalized, G_n is a gradient. However this is unnecessary since we have given in [18] an explicit formula for the functionals F_n, $n=-1, -2, \cdots$, whose gradients G_n satisfy the recursion relation (3.3). They are expressible in terms of the *Floquet exponent* of the operator L, defined as follows:

For any real α, the equation

$$(L-\alpha)w=0$$

has two distinguished solutions $w_{\pm}$ satisfying

$$w_{\pm}(x+p)=K^{\pm 1}w_{\pm}(x).$$

$K=K(\alpha)$ is called the Floquet exponent; it is real in the so-called instability intervals and of modulus one in the stability intervals. Of course K is a functional of u as well as a function of α.

In § 6 of [18] we have shown the following theorem.

THEOREM 3.9. *The gradients of the functionals*

$$F_n=\frac{1}{(-n-1)!}\left(\frac{d}{d\alpha}\right)^{-n-1}\log K(\alpha, u)\bigg|_{\alpha=0}$$

satisfy relations (3.3) *for* $n=-1, -2, \cdots$.

4. Hamiltonian formalism. The Hamiltonian form of equations of motion is

$$\frac{d}{dt}q_j=\frac{\partial H}{\partial p_j}, \qquad \frac{d}{dt}p_j=-\frac{\partial H}{\partial q_j}, \qquad j=1, \cdots, N; \tag{4.1$_H$}$$

the Hamiltonian H is a function of the $2N$ variables $p_j, q_j, j=1, \cdots, N$. Such equations can be cast in another form with the aid of the notion of the *Poisson bracket*, defined for any pair of functions F, K of the $2N$ variables as follows:

$$[F, K]=\sum_j \frac{\partial(F, K)}{\partial(q_j, p_j)}. \tag{4.2}$$

The important properties of the Poisson bracket are:

(a) $[F, K]$ is a bilinear, alternating function of F and K.

(b) The Jacobi identity

$$[[F, H], K]+[[H, K], F]+[[K, F], H]=0. \tag{4.3}$$

In terms of the Poisson bracket the Hamiltonian equations can be expressed as follows:

Let $p(t)$, $q(t)$ be a solution of (4.1), F any function of p, q; then

$$\frac{d}{dt}F(p(t), q(t)) = [F, H]. \tag{4.4}$$

This formulation implies the following theorem.

THEOREM 4.1. *F is a conserved function for all solutions of* $(4.1)_H$ *if and only if*

$$[F, H] = 0. \tag{4.5}$$

It follows from this and the Jacobi identity (4.3) that if F and K are a pair of conserved functions, so is their Poisson bracket. Two functions whose Poisson bracket is zero are said to be *in involution.*

Two further results of Hamiltonian mechanics are the following:

THEOREM 4.2. *Suppose H and K are in involution, i.e.,* $[H, K] = 0$. *Then the Hamiltonian flows generated by H and K, respectively, commute. That is, we denote by $S_H(t)$ and $S_K(t)$ the operator which links initial position to position at time t of a point (p, q) under the Hamiltonian flows* $(4.1)_H$ *and* $(4.1)_K$, *respectively; then*

$$S_H(t)S_K(r) = S_K(r)S_H(t). \tag{4.6}$$

THEOREM 4.3. *Suppose there exist N independent functions $F_1, \cdots, F_N$ in involution which are conserved under the Hamiltonian flow* $(4.1)_H$. *Then* $(4.1)_H$ *is completely integrable.*

5. Hamiltonian structure of KdV and invariant manifolds. C. Gardner in [9] and Faddeev and Zakharov in [6] have independently introduced a Hamiltonian structure for the KdV equation. Here we follow Gardner's line of development.

Gardner introduces the Poisson bracket

$$[F, H] = (G_F, \partial G_H), \tag{5.1}$$

where as before G_F, G_H denote the gradients of F and H with respect to the L_2 scalar product $(\cdot, \cdot)$ and $\partial = d/dx$.

THEOREM 5.1. *The bracket defined by* (5.1) *is*

(a) *bilinear,*

(b) *alternating,*

(c) *satisfies the Jacobi identity.*

We sketch the proof given in [18]. Part (a) is obvious, and part (b) follows by integration by parts. To prove (c) we compute the gradient of $[F, H]$. Using the symmetry of the derivative of G_F and G_H, we can easily show that

$$G_{[F,H]} = N_F\, \partial G_H - N_H\, \partial G_F, \tag{5.2}$$

where N_F and N_H are the second derivatives of F, H. The Jacobi identity (4.3) follows from this if we use once more the symmetry N_F, N_H and N_K.

According to formula (2.3), if $u(t)$ satisfies equation (2.1) and F is any functional,

$$dF/dt = (G_F, K).$$

According to (3.19), for the KdV equation,

$$K = -\partial G_2.$$

Therefore we can write, using (5.1)

$$dF/dt = (G_F, -\partial G_2) = [F, -F_2].$$

Comparing this with (4.4) we see that the KdV equation is of Hamiltonian form, with $H = -F_2$.

Again using (5.1) we can rewrite equations (3.7) as

$$[F_m, F_n] = 0,$$

and equation (3.27) as

$$[\lambda, F_m] = 0.$$

One can show analogously, see [18], that

$$[\lambda, \mu] = 0$$

for any pair of eigenvalues of L. These relations can be expressed by saying *the functionals F_n and λ are in involution.*

Since we don't know an infinite-dimensional analogue of Theorem 4.3, we cannot use this plethora of conserved functions in involution to conclude directly that KdV is integrable. However Theorem 4.2 is true in infinite-dimensional space, and can be used to construct invariant submanifolds of the KdV flow; we outline briefly how:

We start with a variational problem originally suggested by Kruskal and Zabusky:

Given the values

$$F_i(u) = A_i, \qquad i = 0, \cdots, n-1, \tag{5.3}$$

find u which minimizes $(-1)^{N-1}F_N(u)$. The constants A_j have to be so chosen that the constraint (5.3) is satisfied by some function, and so that A_j is not a stationary value of $F_j(u)$ when the other constraints are imposed. This is equivalent to saying that for any function u satisfying the constraints, the gradients

$$G_0(u), G_1(u), \cdots, G_{N-1}(u)$$

are linearly independent. We shall call such constraints *admissible.*

The following theorem was proved in [18].

THEOREM 5.2. *For admissible constraints* (5.3) *the functional* $(-1)^{N-1}F_N(u)$ *is minimized by some function u_0, and every minimizing function satisfies an Euler equation of the form*

$$G(u_0) = 0, \tag{5.4}$$

where

$$G(u) = G_N(u) + \sum_0^{N-1} a_j G_j(u). \tag{5.5}$$

The function G in (5.5) is the gradient of

$$F(u) = F_N(u) + \sum_0^{N-1} a_j F_j(u). \tag{5.6}$$

As we have shown previously,

$$[F_j, F_k] = 0,$$

from which we deduce that

$$[F, F_k] = 0.$$

This implies that F is a conserved functional for all generalized KdV flows

$$u_t = \partial G_k(u). \tag{5.7$_k$}$$

Denote by $S_k(t)$ the solution operator for equation $(5.7)_k$. It follows from Theorem 2.2 that if u_0 satisfies equation (5.4), then so does every function of the form

$$u = \prod_1^{N-1} S_k(t_k)u_0, \qquad -\infty < t_k < \infty. \tag{5.8}$$

Denote the set (5.8) by $S = S(A_0, \cdots, A_{N-1})$; the following result was proved in [18].

THEOREM 5.3. *S is a compact $(N-1)$-dimensional manifold.*

The operators $S_k(t)$ map S onto itself, and they commute. Denote by Ω the collection of those vectors $(\omega_1, \cdots, \omega_{N-1})$ for which

$$\prod S_k(\omega_k) = I.$$

Ω is the module of periods. It follows from the definition of S that

$$S = \mathbb{R}^M/\Omega.$$

Since S is $(N-1)$-dimensional and compact, it follows that Ω is a lattice (i.e., discrete) and $(N-1)$-dimensional; from this we conclude the following theorem.

THEOREM 5.4. *S is an $(N-1)$-dimensional torus, and each $S_k(t)$ is almost periodic on S.*

Every point on $S(A_0, \cdots, A_{N-1})$ minimizes $F_N(u)$ subject to the constraints (5.3). There may be points not on S which minimize $F_N(u)$; but it follows from the above analysis that the minimizing set is a union of disjoint $(N-1)$-dimensional tori.

The case $N=1$ is trivial and the case $N=2$ is classical, going back to Korteweg–de Vries [15].

Using formulas (3.2) we get the following expression for equation (5.4), $N=2$:

$$u_{xx} + \tfrac{1}{2}u^2 + a_1 u + a_0 = 0.$$

Multiplying this by $2u_x$ we obtain an equation of the form

$$u_x^2 = Q(u),$$

where Q is a cubic polynomial. From this we can express x as function of u by an elliptic integral, which shows that u_0 is an elliptic function of x. It can be shown that in this case the minimizing set is a single circle formed by the translates of an elliptic function u_0. It is easy to verify that the function

$$u_0(x + a_1 t)$$

is a solution of the KdV equation. This traveling wave has been called "cnoidal wave" by Korteweg and de Vries.

Benjamin [2] and Bona [4] have proved the stability of simple cnoidal waves; i.e., they have shown that if $u_1(s)$ is sufficiently near $u_0(x)$ in an appropriate metric, then for any value of t, $S_2(t)u_1$, the solution of KdV with initial value u_1, lies near $u_0(x+\theta)$ for some θ.

We surmise that all solutions $S_2(t)u_0$ constructed in this section are stable in the above sense, as long as u_0 is an absolute—or even just local—minimum of $F_N(u)$ among all u satisfying constraints (5.3). The numerical experiments carried out by M. Hyman and described in an Appendix to this paper certainly very strongly suggest this.

Next we sketch a simple argument which shows that F_N has stationary points on the constrained set (5.3) which are not minima. We use Morse theory, according to which such stationary points exist if the homology of the set (5.3) is not trivial.

THEOREM 5.5. *For suitably chosen constants* $A_0, \cdots, A_{N-1}$ *the constrained set* (5.3) *is not simply connected.*

Proof. We shall handle the case $N=3$. A_0 and A_1 are taken as arbitrary, and we denote by M the minimum of $-F_N$ on the set (5.3), $N=2$. As remarked earlier, in this case the minimizing set is a single circle consisting of all translates of an elliptic function $u_0(x)$.

Let n be an index $\neq 0$ for which the nth Fourier coefficient of u_0 is $\neq 0$:

$$\int u_o(x)\, e^{-inx}\, dx \neq 0. \tag{5.9}$$

Consider the set of functions u which satisfy the constraints (5.3), $N=2$ and the additional constraint

$$\int u(x)\, e^{-inx}\, dx = 0. \tag{5.9$'$}$$

Since this constraint excludes the solutions of the previous minimum problem, it follows that the minimum value M_1 of $-F_2(u)$ subject to this new constraint exceeds the old minimum M of $-F_2(u)$:

$$M < M_1.$$

We choose now a function u_1 different from u_0 but so close to it that

$$-F_2(u_1) < M_1. \tag{5.10}$$

We claim that the circle

$$\theta \to u_1(x+\theta)$$

cannot be deformed to a point on the set of those u which satisfy

$$F_j(u) = F_j(u_1), \qquad j = 0, 1, 2. \tag{5.11}$$

To see this we introduce the projection P onto the nth Fourier coefficient. If $u_1(x, \theta, s)$, $0 \leqq s \leqq 1$, were a deformation of $u_1(x+\theta)$ to a point, then $Pu_1(x, \theta, s)$ would be a deformation of $a_n\, e^{-in\theta}$ to a point in the complex plane. Since $a_n\, e^{-in\theta}$, $n \neq 0$, winds around the origin, such a deformation would have to cross the origin; but then for some value of θ and s the nth Fourier coefficient of the function $u = u(x+\theta, s)$ is zero. This implies by (5.9) that

$$-F_2(u) \geqq M_1,$$

which when combined with (5.11) contradicts (5.10).

Having proved the existence of a curve in the set (5.11) which is not homotopic zero we consider all curves C in the same homotopy class, and determine that one for which the maximum of $F_3(u)$ on C is as small as possible. I surmise that this minimax problem has a solution; such a solution satisfies an equation of the form (5.4), (5.5). As in the case of the minimum problem, the solutions of the minimax problem form a 2-dimensional torus.

I suspect the solutions of equations of the form (5.4), (5.5), with a_j and N arbitrary, are dense among all C^∞ periodic functions.

The author and Jurgen Moser have shown, see [17] and [18], that if u satisfies an equation of the form (5.4), then all but a finite number of eigenvalues of the Schrödinger operator L defined by (3.22) are double. Using this connection, and a method of Hochstadt [14], McKean and van Moerbeke were able to use the inverse method in spectral theory to study periodic solutions of (5.4). They have given a new proof of Theorem 5.4, and were able to express solutions of (5.4) as hyperelliptic functions. There is hope that their approach can be used to settle the question of integrability of the KdV equation in the class of periodic functions.

Very recently McKean and Trubowitz succeeded in showing that all solutions of KdV which are periodic in x are indeed almost periodic in t.

APPENDIX

JAMES M. HYMAN

In this Appendix we describe how the construction of special solutions of the KdV equation which minimize F_2 subject to the constraint $F_j = A_j$ can be implemented numerically. We saw in § 3 that a minimizing function satisfies the Euler equation

$$G_3 + \sum_0^2 a_j G_j = 0. \tag{A.1}$$

A solution of (A.1) is an extremal for

$$F = F_3 + \sum_0^2 a_j F_j, \tag{A.2}$$

and presumably can be obtained by minimizing that functional. This is indeed what we did. We chose the constants a_{-1}, a_0, a_1, then discretized the functional (A.2) by specifying u at N equidistant points and expressed the first and second derivatives of u in (A.2) by difference quotients. The resulting function of N variables was minimized using A. Jameson's version[1] of the Fletcher–Powell–Davidon algorithm [8]. The resulting discretized function is a somewhat crude

[1] My thanks are due to A. Jameson for acquainting me with his FPD package.

approximation to the function u we are looking for. To make u more accurate we proceeded as follows:

We are looking for a periodic solution of (A.1); since this is a fourth order equation, its solutions are parametrized by their four Cauchy data at, say, $x=0$. Periodicity requires the Cauchy data at $x=p$ be equal to the Cauchy data at $x=0$. We sought to satisfy this requirement by a sequence of approximations constructed by "shooting." The Cauchy data of u_{n+1} at $x=0$ was chosen by applying the rule of false position to $u_n(p)-u_n(0)$, $u_n'(p)-u_n'(0)$, $u_{n-1}(p)-u_{n-1}(0)$ and $u_{n-1}'(p)-u_{n-1}'(0)$. $u_{n+1}(p)$ and $u_{n+1}'(p)$ were then computed numerically. For this purpose we used the ODE package developed by A. Hindmarsh [13].

The following observations were helpful:

(a) Instead of matching all four Cauchy data for the fourth order equation (A.1), it sufficed to match only two, since according to § 5 periodic solutions form a two-parameter family, and therefore two of the matching conditions must be consequences of the other two.

(b) The initial guesses for the Cauchy data come from the approximate solution obtained by the variational procedure. Without such a good initial guess we were unable to construct a periodic solution of (A.1).

The periodic solution of (A.1) constructed above was then used as initial value data. We solved numerically the initial value problem for the KdV equation using Fred Tappert's[2] method [26] and code. The accuracy of the solution was monitored by checking the constancy of the functionals F_0, F_1, F_2, and the extent

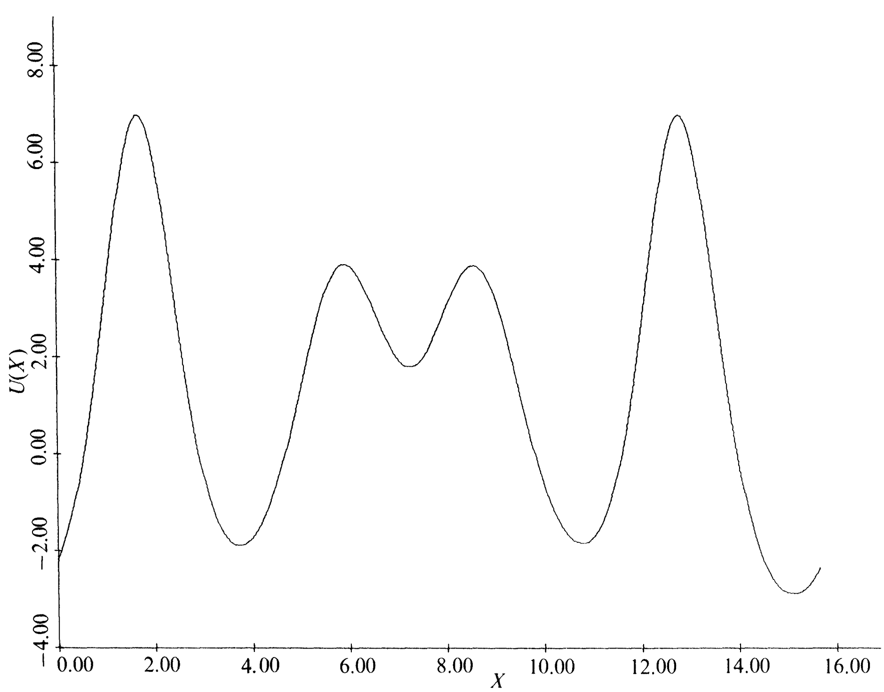

FIG. 1. *Periodic solution to equation* (*A*.1) *with* $a_0=0$, $a_1=-8$, $a_2=2$ *and period* $=15.7$

[2] My thanks are due to Fred Tappert for acquainting me with his KdV solver.

to which the solution satisfied the ODE (A.1). The solution constructed by Tappert's method passed these tests of accuracy reasonably well. Solutions constructed using earlier methods of Kruskal and Zabusky [29] and Vliegenthart [28] were less accurate and were not used in this study. The finest vindication of Tappert's KdV solver was that after a finite elapse of time the solution resumed its initial shape, in a shifted position. During the intervening time the shape of the solution underwent considerable gyrations.

In Fig. 1 we present a periodic solution of equation (A.1), with $a_0 = 0$, $a_1 = -8$, $a_2 = 2$ and period $= 15.7$. Figures 2 to 5 show the value at times $t = .28$, $.56$, $.84$, and 1.12 of the solution to the KdV equation with initial values shown in Fig. 1. Note that the function shown in Fig. 5 has the same shape as the initial function in Fig. 1, except for a shift by the amount -5.1.

Figures 6 to 10 show the time history of a solution of KdV where the initial value was obtained by superimposing a random disturbance[3] on the initial function shown in Fig. 1. Note that the disturbance is not magnified during the flow, and that the averages of these disturbed signals are very close at all times shown to the undisturbed signals pictured in Figs. 2–5. This calculation demonstrates convincingly the great stability of the KdV flow pictured here. It also demonstrates the ability to Tappert's KdV solver to deal accurately with solutions containing high frequency disturbances.

Figure 11 shows a double cnoidal wave over two periods. The solution of KdV with this initial value propagates with speed $c = .3$ without altering its shape. Figure 12 shows a function obtained from Fig. 11 by raising the first peak by 20% and lowering the second by 20%. Figure 13 shows the solution at time $t = 0.52$ using the function in Fig. 12 as initial data for the KdV equation. At time $t = 1.04$ the solution returns to its original shape in Fig. 12 while translating with speed $c \approx .3$. This and other calculations indicate that the double cnoidal wave is stable.

In [3], Benjamin presents an elegant argument to show that the second variation of $-F_2$ under constraint of F_0 and F_1 is indefinite for a double cnoidal wave, and therefore a double cnoidal wave is not even a local minimum of $-F_2$ under the constraints; he raises the question whether this implies instability of the double cnoidal wave. The numerical study reported above and others unreported here indicate stability. We remark that this is not surprising from the point of view of Hamiltonian theory.

[3] We learned recently that in 1965 Zabusky tested the stability of solutions of the KdV equation by imposing random disturbances on their initial data; he too observed that solutions were remarkably stable under such perturbations.

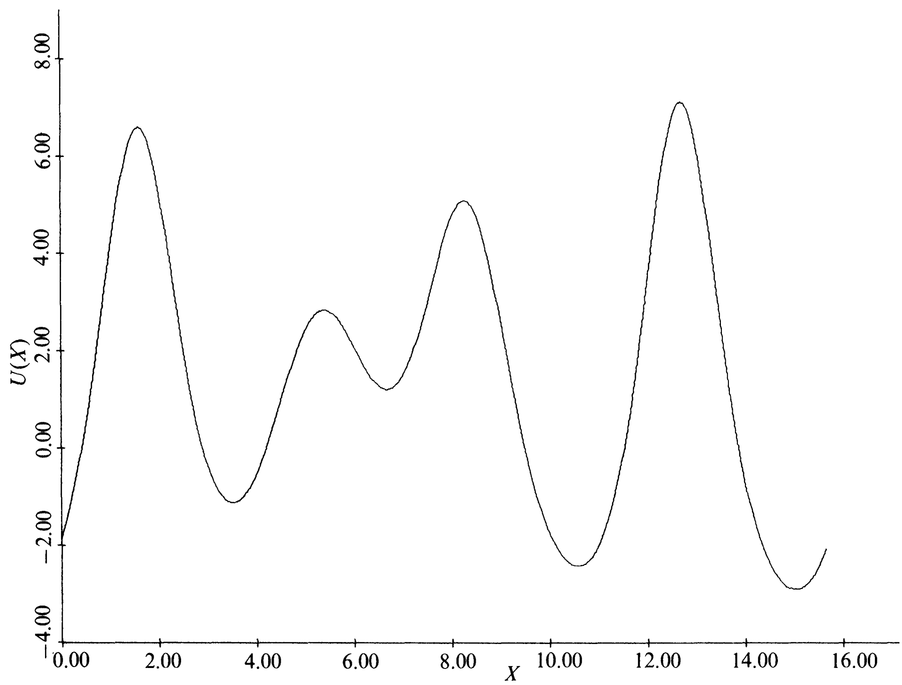

FIG. 2. *The solution of the KdV equation at time $t = .28$ with initial value shown in Fig.* 1

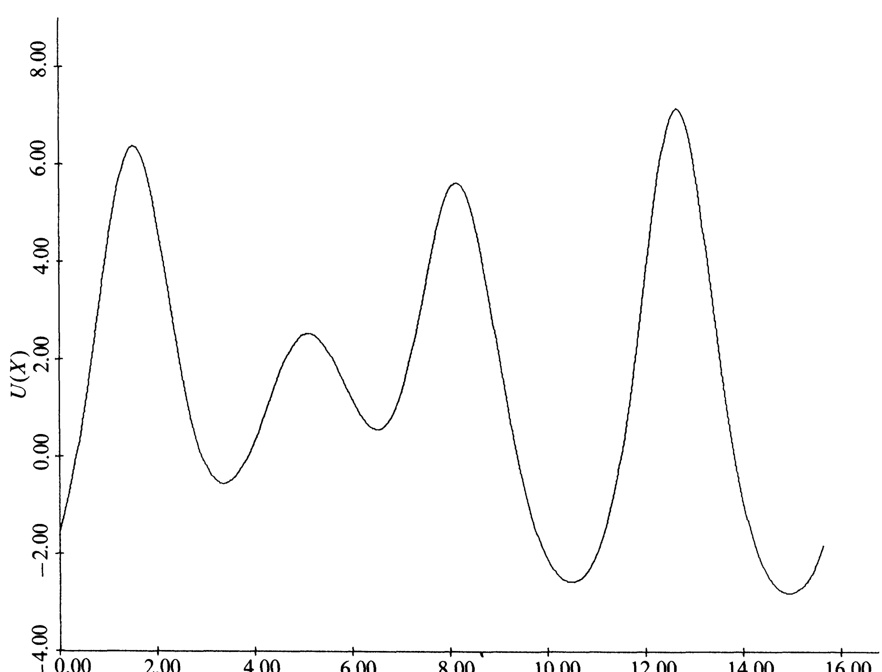

FIG. 3. *The solution of the KdV equation at time $t = .56$ with initial value shown in Fig.* 1

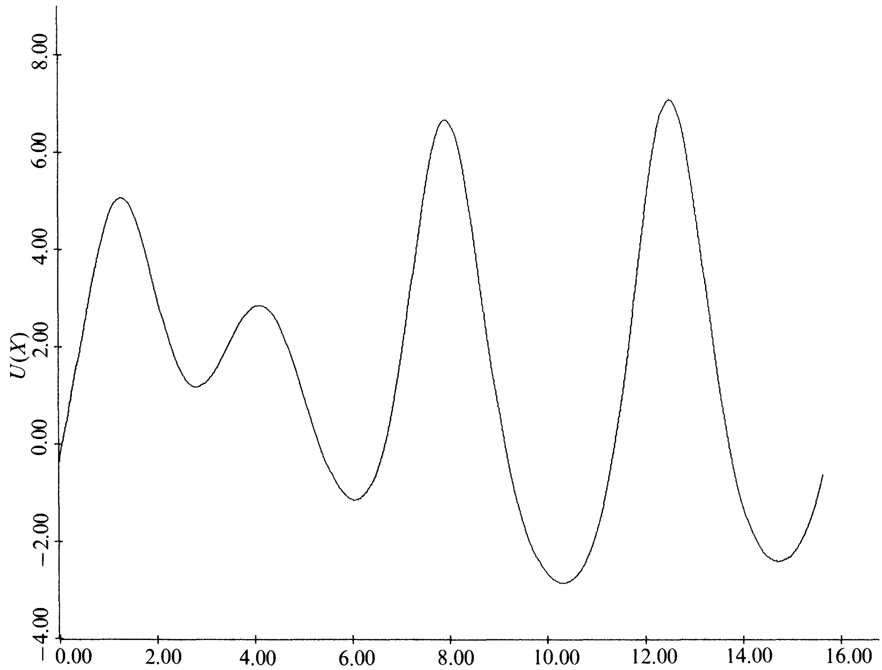

FIG. 4. *The solution of the KdV equation at time* $t = .84$ *with initial value shown in Fig.* 1

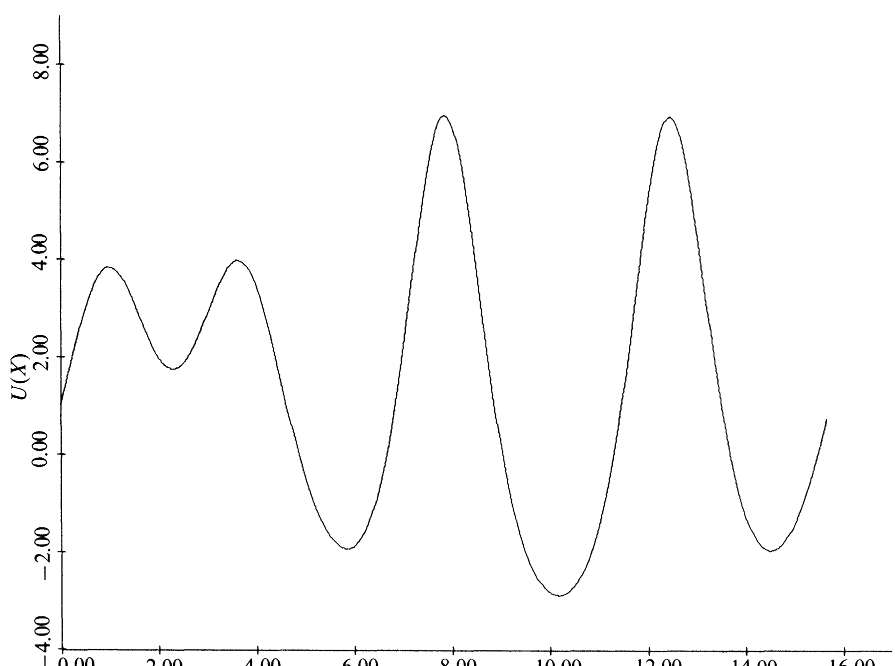

FIG. 5. *The solution of the KdV equation at time* $t = 1.12$ *with the initial value shown in Fig.* 1

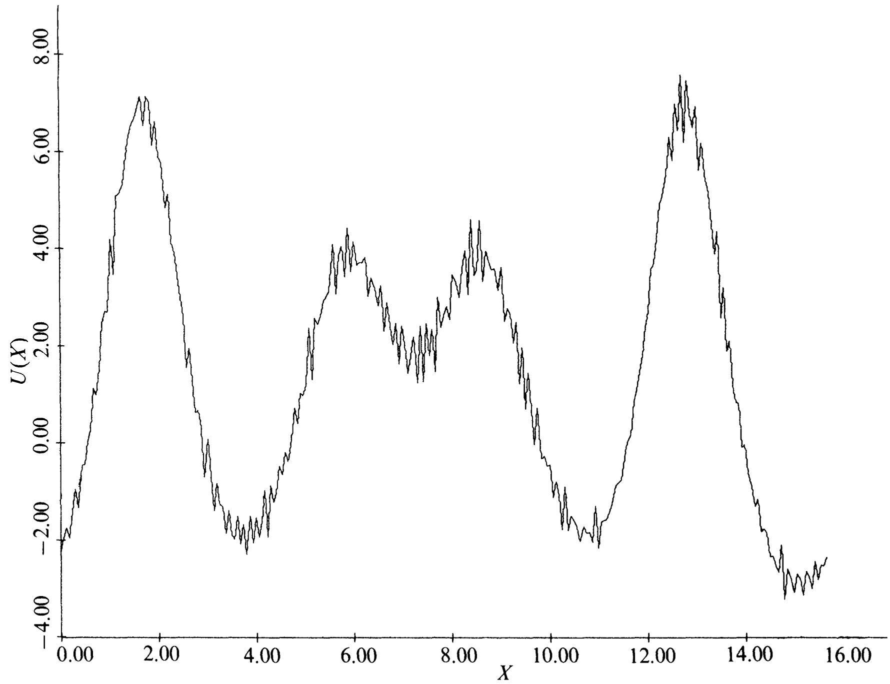

FIG. 6. *A random disturbance is superimposed on the solution shown in Fig.* 1

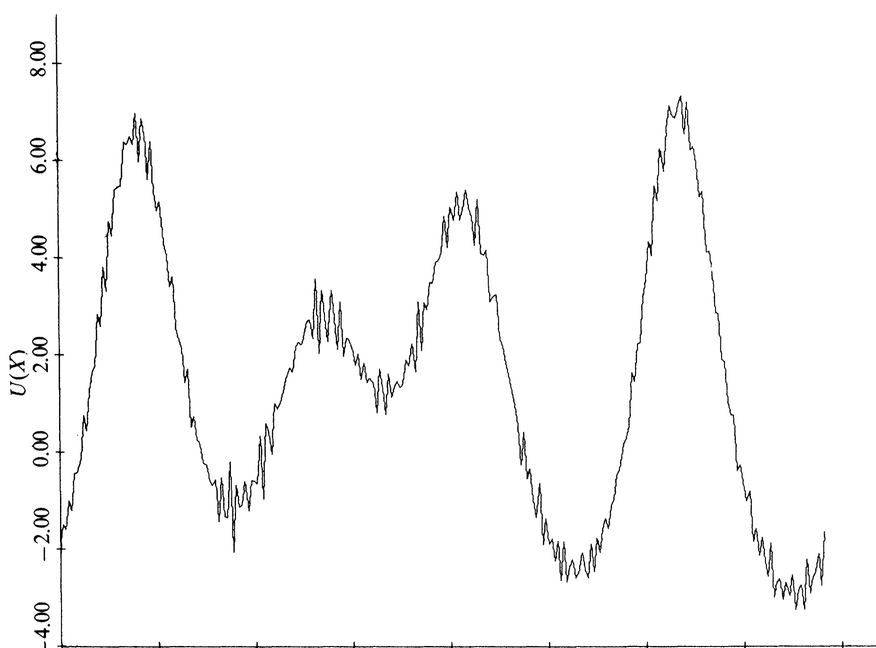

FIG. 7. *The solution of the KdV equation at time* $t = .28$ *with initial value shown in Fig.* 6

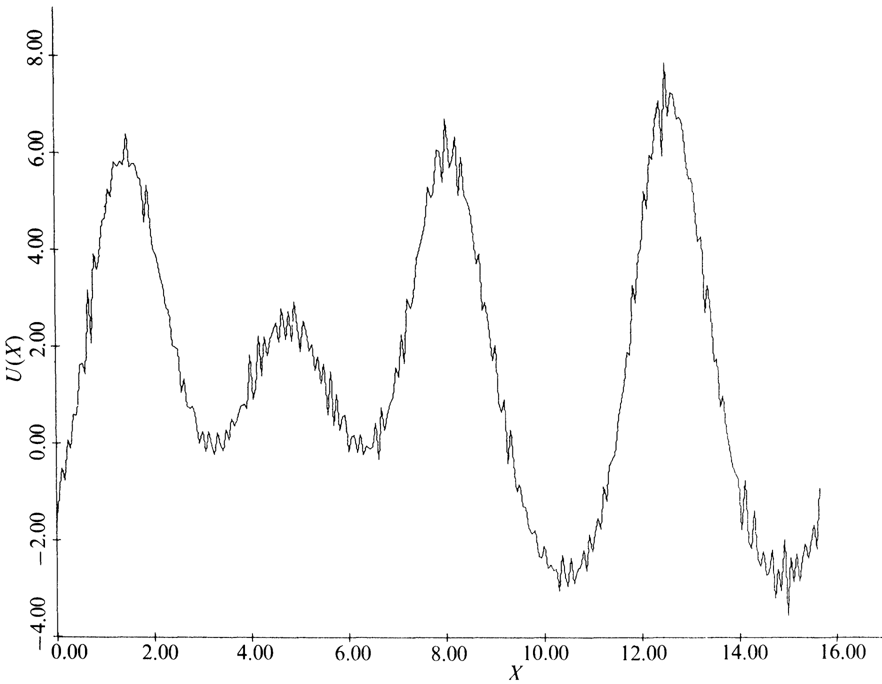

FIG. 8. *The solution of the KdV equation at time $t = .56$ with initial value shown in Fig.* 6

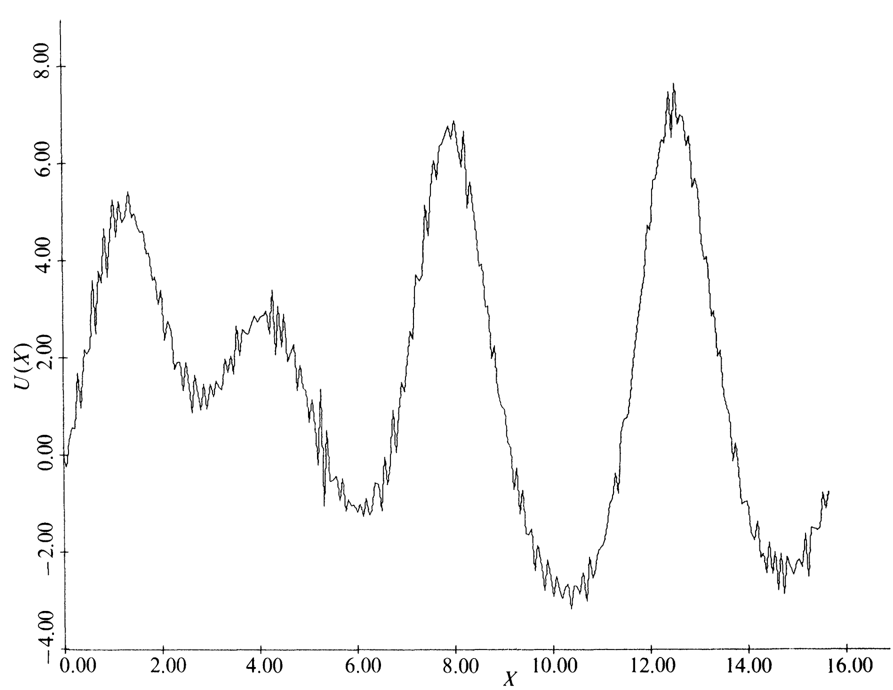

FIG. 9. *The solution of the KdV equation at time $t = .84$ with initial value shown in Fig.* 6

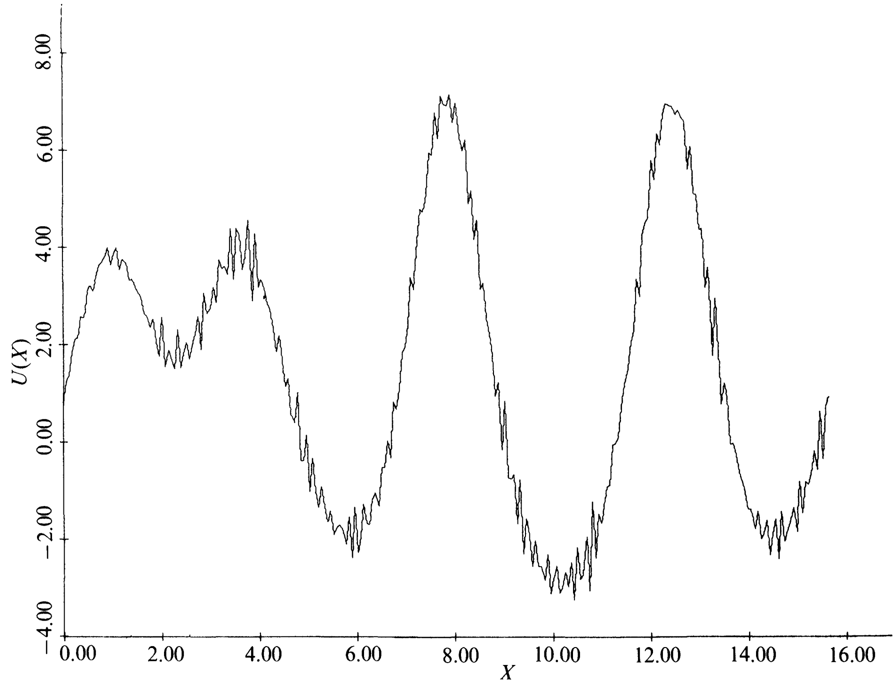

FIG. 10. *The solution of the KdV equation at time* $t = 1.12$ *with the initial value shown in Fig.* 6

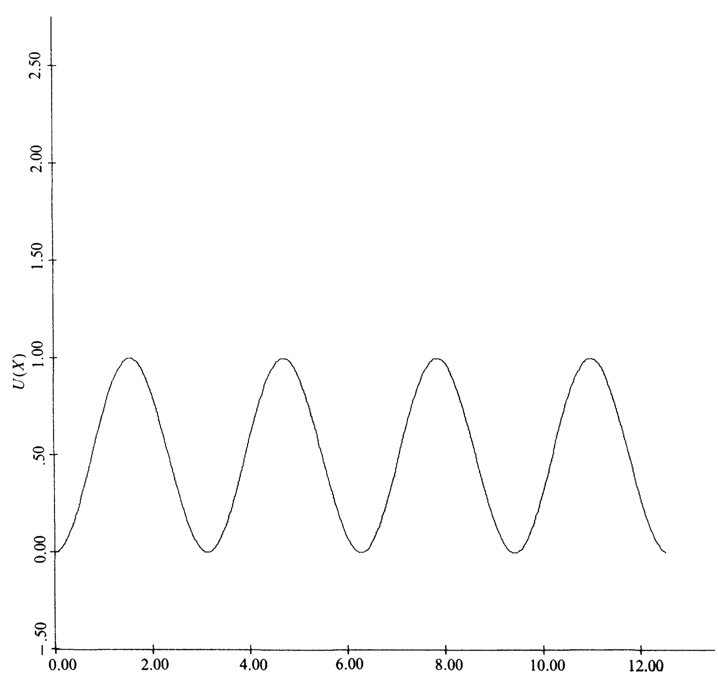

FIG. 11. *A double cnoidal wave over two periods*

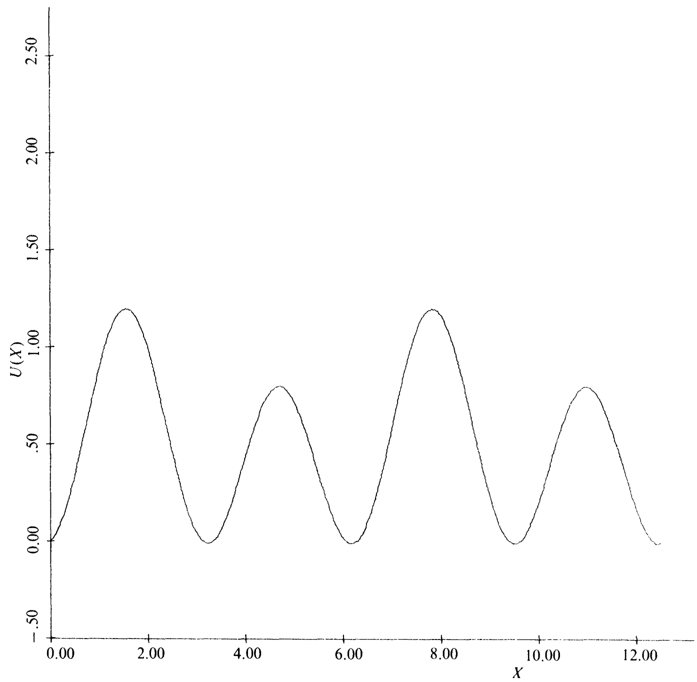

FIG. 12. *The first peak of the cnoidal wave in Fig.* 11 *was raised by* 20% *and the second was lowered by* 20%

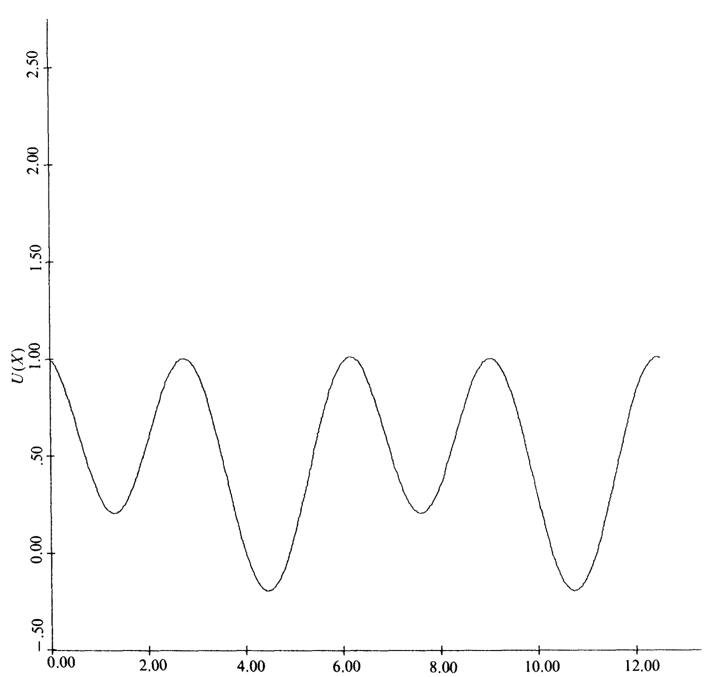

FIG. 13. *The solution to the KdV equation at* $t = .52$ *using the function shown in Fig.* 12 *as initial data. At* $t = 1.04$ *the solution returns to approximately the shape in Fig.* 12

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